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Fixed points and endpoints of contractive set-valued maps in cone uniform spaces with generalized pseudodistances

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Abstract

We introduce the concept of contractive set-valued maps in cone uniform spaces with generalized pseudodistances and we show how in these spaces our fixed point and endpoint existence theorem of Caristi type yields the fixed point and endpoint existence theorem for these contractive maps.

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1 Introduction

Nadler [1] extended Banach's fixed point theorem [2] for set-valued maps in complete metric spaces.

Theorem 1.1 ([1, Th. 5]) *Let (X,d) be a complete metric space, let $\mathrm{Cl}(X)$ denote the class of all nonempty closed subsets of X, and let $H:(\mathrm{Cl}(X))^2 \to [0,\infty]$ be defined by*

$$\forall_{A,B\in \mathrm{Cl}(X)}\Big\{H(A,B) = \max\Big\{\sup_{u\in A} d(u,B), \sup_{v\in B} d(v,A)\Big\}\Big\},$$

where, for each $u\in X$ and $V\in \mathrm{Cl}(X)$, $d(u,V) = \inf_{v\in V} d(u,v)$. If a set-valued map $T: X\to \mathrm{Cl}(X)$ is H-contractive, i.e., if T satisfies

$$\exists_{0<\lambda<1}\forall_{x,y\in X}\big\{H\big(T(x),T(y)\big)\le \lambda d(x,y)\big\},$$

then T has a fixed point w in X, i.e., $w\in T(w)$.

A number of authors introduce the new concepts of set-valued contractions of Nadler type and study the problem concerning the existence of fixed points for such contractions; see, *e.g.*, Aubin and Siegel [3], de Blasi *et al.* [4], Ćirić [5], Eldred *et al.* [6], Feng and Liu [7], Frigon [8], Al-Homidan *et al.* [9], Jachymski [10], Kaneko [11], Klim and Wardowski [12], Latif and Al-Mezel [13], Mizoguchi and Takahashi [14], Pathak and Shahzad [15], Quantina and Kamran [16], Reich [17, 18], Reich and Zaslavski [19, 20], Sintunavarat and Kumam [21–25], Suzuki [26], Suzuki and Takahashi [27], Takahashi [28] and Zhong *et al.* [29]. In particular, the significant fixed point existence results of Nadler type were obtained by Suzuki [30, Th. 3.7] in metric spaces with τ-distances and by Wardowski [31] in cone metric spaces.

Recently, Włodarczyk and Plebaniak in [32] have studied among others the $\mathcal{J}$-families of generalized pseudodistances in cone uniform, uniform and metric spaces which generalize distances of Tataru [33], w-distances of Kada *et al.* [34], τ-distances of Suzuki [35] and τ-functions of Lin and Du [36] in metric spaces and distances of Vályi [37] in uniform spaces.

In the present paper, we introduce the concept of contractive set-valued maps in cone uniform spaces with generalized pseudodistances, and we show how in these spaces our fixed point and endpoint existence theorem of Caristi type [32, Th. 4.5] yields the fixed point and endpoint existence theorem for these contractive maps.

It is worth noticing that our fixed point and endpoint existence Theorem 3.1: has a simpler proof; is Nadler type; is new in cone uniform and cone locally convex spaces; is new even in cone metric and metric spaces; and is different from those given in the previous publications on this subject.

This paper is a continuation of [32, 38–46].

2 Definitions and notations

We define a *real normed space* to be a pair $(L, \|\cdot\|)$ with the understanding that a vector space L over $\mathbb{R}$ carries the topology generated by the metric $(a,b) \to \|a-b\|$, $a, b \in L$.

A nonempty closed convex set $H \subset L$ is called a *cone* in L if it satisfies:

(H1) $\forall_{s\in(0,\infty)}\{sH \subset H\}$;

(H2) $H \cap (-H) = \{0\}$; and

(H3) $H \neq \{0\}$.

It is clear that each cone $H \subset L$ defines, by virtue of

"$a \preceq_H b$ iff $b - a \in H$",

an *order* of L under which L is an *ordered normed space* with a cone H.

We will write $a \prec_H b$ to indicate that $a \preceq_H b$, but $a \neq b$. A cone H is said to be *solid* if $\operatorname{int}(H) \neq \emptyset$; $\operatorname{int}(H)$ denotes the interior of H. We will write $a \ll_H b$ to indicate that $b - a \in \operatorname{int}(H)$.

The cone H is *normal* if a real number $M > 0$ such that for each $a, b \in H$, $0 \preceq_H a \preceq_H b$ implies $\|a\| \leq M\|b\|$ exists. The number M satisfying above is called the *normal constant* of H.

Let an element $+\infty \notin L$ be such that $a \preceq_H +\infty$ for all $a \in L$.

Let 2^X denote the family of all nonempty subsets of a space X. Recall that a *set-valued dynamic system* is defined as a pair (X, T), where X is a certain space and T is a set-valued map $T : X \to 2^X$; in particular, a set-valued dynamic system includes the usual dynamic system where T is a single-valued map. We say that a map $\omega : X \to L \cup \{+\infty\}$ is *proper* if its effective domain, $\operatorname{dom}(\omega) = \{x : \omega(x) \neq +\infty\}$, is nonempty.

Definition 2.1 ([38, Def. 2.2]) Let X be a nonempty set, and let L be an ordered normed space with a cone H.

(i) The family $\mathcal{P} = \{p_\alpha : X^2 \to L, \alpha \in \mathcal{A}\}$, $\mathcal{A}$-index set, is said to be a *$\mathcal{P}$-family of cone pseudometrics on X* (*$\mathcal{P}$-family* for short) if the following three conditions hold:

($\mathcal{P}1$) $\forall_{\alpha\in\mathcal{A}}\forall_{x,y\in X}\{0 \preceq_H p_\alpha(x,y) \wedge x = y \Rightarrow p_\alpha(x,y) = 0\}$;
($\mathcal{P}2$) $\forall_{\alpha\in\mathcal{A}}\forall_{x,y\in X}\{p_\alpha(x,y) = p_\alpha(y,x)\}$; and
($\mathcal{P}3$) $\forall_{\alpha\in\mathcal{A}}\forall_{x,y,z\in X}\{p_\alpha(x,z) \preceq_H p_\alpha(x,y) + p_\alpha(y,z)\}$.

(ii) If $\mathcal{P}$ is a $\mathcal{P}$-family, then the pair $(X,\mathcal{P})$ is called a *cone uniform space*.
(iii) A $\mathcal{P}$-family $\mathcal{P}$ is said to be *separating* if

($\mathcal{P}4$) $\forall_{x,y\in X}\{x \neq y \Rightarrow \exists_{\alpha\in\mathcal{A}}\{0 \prec_H p_\alpha(x,y)\}\}$.

(iv) If a $\mathcal{P}$-family $\mathcal{P}$ is separating, then the pair $(X,\mathcal{P})$ is called a *Hausdorff cone uniform space*.

Definition 2.2 ([38, Def. 2.3]) Let L be an ordered normed space with a solid cone H, and let $(X,\mathcal{P})$ be a Hausdorff cone uniform space with a cone H.

(i) We say that a sequence $(w_m : m \in \mathbb{N})$ in X is a *$\mathcal{P}$-convergent in X* (*convergent in X* for short) if there exists $w \in X$ such that

$$\forall_{\alpha\in\mathcal{A}}\forall_{c_\alpha\in L,0\ll c_\alpha}\exists_{n_0=n_0(\alpha,c_\alpha)\in\mathbb{N}}\forall_{m\in\mathbb{N};n_0\leq m}\big\{p_\alpha(w_m,w) \ll_H c_\alpha\big\}.$$

(ii) We say that a sequence $(w_m : m \in \mathbb{N})$ in X is a *$\mathcal{P}$-Cauchy sequence in X* (*Cauchy sequence in X*, for short) if

$$\forall_{\alpha\in\mathcal{A}}\forall_{c_\alpha\in L,0\ll c_\alpha}\exists_{n_0=n_0(\alpha,c_\alpha)\in\mathbb{N}}\forall_{m,n\in\mathbb{N};n_0\leq m<n}\big\{p_\alpha(w_m,w_n) \ll_H c_\alpha\big\}.$$

(iii) If every Cauchy sequence in X is convergent in X, then $(X,\mathcal{P})$ is called a *$\mathcal{P}$-sequentially complete cone uniform space* (*sequentially complete* for short).

Theorem 2.1 ([32, Th. 2.3]) *Let L be an ordered Banach space with a normal solid cone H, and let $(X,\mathcal{P})$ be a Hausdorff cone uniform space with a cone H. The following hold*:

(P1) *The sequence $(w_m : m \in \mathbb{N})$ in X converges to $w \in X$ iff*

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon_\alpha>0}\exists_{n_0\in\mathbb{N}}\forall_{m\in\mathbb{N};m\geq n_0}\big\{\big\|p_\alpha(w_m,w)\big\| < \varepsilon_\alpha\big\}.$$

(P2) *The sequence $(w_m : m \in \mathbb{N})$ in X is a Cauchy sequence in X iff*

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon_\alpha>0}\exists_{n_0\in\mathbb{N}}\forall_{m,n\in\mathbb{N};m>n\geq n_0}\big\{\big\|p_\alpha(w_m,w_n)\big\| < \varepsilon_\alpha\big\}.$$

Definition 2.3 Let L be an ordered Banach space with a cone H.

(i) A subset $D \subset L$ is said to have a *minimal* (*maximal*) *element* if there exists $a \in D$ such that $a \preceq_H b$ ($b \preceq_H a$) for all $b \in D$, and we write then that $a = \min(D)$ ($a = \max(D)$). It is clear that if D has a minimal (maximal) element, then the minimal (maximal) element is unique.
(ii) We say that $a \in L$ is an *infimum* (*supremum*) for set $D \subset L$ if $\mathrm{cl}_L(D)$ has the minimal (maximal) element and $a = \min(\mathrm{cl}_L(D))$ ($a = \max(\mathrm{cl}_L(D))$), and we write then that $a = \inf(D)$ ($a = \sup(D)$); here $\mathrm{cl}_L(D)$ denotes the closure of D in L.

Definition 2.4 Let L be an ordered normed space with a solid cone H. The cone H is called *regular* if for every increasing (decreasing) sequence $(c_m : m \in \mathbb{N})$ in L which is bounded from above (below),

$$\big(i.e., c_1 \preceq_H c_2 \preceq_H \cdots \preceq_H c_m \preceq_H \cdots \preceq_H b$$
$$(b \preceq_H \cdots \preceq_H c_m \preceq_H \cdots \preceq_H c_2 \preceq_H c_1) \text{ for some } b \in L\big),$$

there exists $c \in L$ such that $\lim_{m\to\infty} \|c_m - c\| = 0$. Every regular cone is normal.

Definition 2.5 ([32, Def. 2.6]) Let L be an ordered normed space with a normal solid cone H, and let $(X, \mathcal{P})$ be a Hausdorff cone uniform space with a cone H.

(i) The family $\mathcal{J} = \{J_\alpha : X^2 \to L, \alpha \in \mathcal{A}\}$ is said to be a *$\mathcal{J}$-family of cone pseudodistances on X* (*$\mathcal{J}$-family on X* for short) if the following three conditions hold:

($\mathcal{J}$1) $\forall_{\alpha\in\mathcal{A}}\forall_{x,y\in X}\{0 \preceq_H J_\alpha(x,y)\}$;
($\mathcal{J}$2) $\forall_{\alpha\in\mathcal{A}}\forall_{x,y,z\in X}\{J_\alpha(x,z) \preceq_H J_\alpha(x,y) + J_\alpha(y,z)\}$; and
($\mathcal{J}$3) For any sequence $(w_m : m \in \mathbb{N})$ in X such that

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon_\alpha>0}\exists_{n_0=n_0(\alpha,\varepsilon_\alpha)\in\mathbb{N}}\forall_{m,n\in\mathbb{N};n_0\le m\le n}\big\{\big\|J_\alpha(w_m,w_n)\big\| < \varepsilon_\alpha\big\},$$

if there exists a sequence $(v_m : m \in \mathbb{N})$ in X satisfying

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon_\alpha>0}\exists_{n_0=n_0(\alpha,\varepsilon_\alpha)\in\mathbb{N}}\forall_{m\in\mathbb{N};n_0\le m}\big\{\big\|J_\alpha(w_m,v_m)\big\| < \varepsilon_\alpha\big\},$$

then

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon_\alpha>0}\exists_{n_0=n_0(\alpha,\varepsilon_\alpha)\in\mathbb{N}}\forall_{m\in\mathbb{N};n_0\le m}\big\{\big\|p_\alpha(w_m,v_m)\big\| < \varepsilon_\alpha\big\}.$$

(ii) Each $\mathcal{P}$-family is a $\mathcal{J}$-family.

(iii) If $\mathcal{J} = \{J_\alpha : X^2 \to L : \alpha \in \mathcal{A}\}$ is a $\mathcal{J}$-family, then $X = X^0_{\mathcal{J}} \cup X^+_{\mathcal{J}}$ where

$$X^0_{\mathcal{J}} = \big\{x \in X : \forall_{\alpha\in\mathcal{A}}\big\{0 = J_\alpha(x,x)\big\}\big\}$$

and

$$X^+_{\mathcal{J}} = \big\{x \in X : \exists_{\alpha\in\mathcal{A}}\big\{0 \prec_H J_\alpha(x,x)\big\}\big\}.$$

Let $(X, \mathcal{P})$ be a sequentially complete cone uniform space. We say that a set $Y \in 2^X$ is *closed* in X if $Y = \mathrm{cl}_X(Y)$ where $\mathrm{cl}_X(Y)$, the *closure* of Y in X, denotes the set of all $w \in X$ for which there exists a sequence $(w_m : m \in \mathbb{N})$ in Y which converges to w. If a set $Y \in 2^X$ is closed in X, then $(Y, \mathcal{P})$ is a sequentially complete cone uniform space with a cone H. Define $\mathrm{Cl}(X) = \{Y \in 2^X : Y = \mathrm{cl}_X(Y)\}$; that is, $\mathrm{Cl}(X)$ denotes the class of all nonempty closed subsets of X.

Definition 2.6 Let L be an ordered Banach space with a normal solid cone H, let $(X, \mathcal{P})$ be a Hausdorff sequentially complete cone uniform space with a cone H, and let $\mathcal{J} = \{J_\alpha : X^2 \to L, \alpha \in \mathcal{A}\}$ be a $\mathcal{J}$-family.

(i) Let $A, B \in \mathrm{Cl}(X)$. We say that a pair (A, B) is *$\mathcal{J}$-admissible* if:

(a) For each $\alpha \in \mathcal{A}$, $x \in A$ and $y \in B$, the set $\mathrm{cl}_L(\{J_\alpha(x,v) : v \in B\})$ has a minimal element, say $J_\alpha(x,B)$ (*i.e.*, $J_\alpha(x,B) = \inf_{v\in B} J_\alpha(x,v)$), and the set $\mathrm{cl}_L(\{J_\alpha(y,u) : u \in A\})$ has a minimal element, say $J_\alpha(y,A)$ (*i.e.*, $J_\alpha(y,A) = \inf_{u\in A} J_\alpha(y,u)$);

(b) The sets $\mathrm{cl}_L(\{J_\alpha(u,B) : u \in A\})$ and $\mathrm{cl}_L(\{J_\alpha(v,A) : v \in B\})$ have maximal elements, say $J_\alpha(A,B)$ and $J_\alpha(B,A)$, respectively (*i.e.*,

$$J_\alpha(A,B) = \sup_{u\in A} J_\alpha(u,B) = \sup_{u\in A} \inf_{v\in B} J_\alpha(u,v)$$

and

$$J_\alpha(B,A) = \sup_{v\in B} J_\alpha(v,A) = \sup_{v\in B} \inf_{u\in A} J_\alpha(v,u),$$

respectively); and

(c) For each $\alpha \in \mathcal{A}$, the elements $J_\alpha(A,B)$ and $J_\alpha(B,A)$ are comparable.

(ii) Let $A, B \in \mathrm{Cl}(X)$, and let a pair (A,B) be $\mathcal{J}$-admissible. For each $\alpha \in \mathcal{A}$, we define $\mathcal{H}^{\mathcal{J}} = \{H^{\mathcal{J}}_\alpha(A,B), \alpha \in \mathcal{A}\}$ where

$$\forall_{\alpha\in\mathcal{A}} \bigl\{ H^{\mathcal{J}}_\alpha(A,B) = \max\bigl\{J_\alpha(A,B), J_\alpha(B,A)\bigr\}\bigr\}.$$

Here, for each $\alpha \in \mathcal{A}$, $H^{\mathcal{J}}_\alpha(A,B) \in L \cup \{+\infty\}$ and by $(\mathcal{J}1)$ and since H is closed, $0 \preceq_H H^{\mathcal{J}}_\alpha(A,B)$.

(iii) Let a set-valued dynamic system (X,T) satisfy $T : X \to \mathrm{Cl}(X)$. We say that (X,T) is $\mathcal{J}$*-admissible* if for each $x, y \in X$, a pair $(T(x), T(y))$ is $\mathcal{J}$-admissible.

(iv) Let (X,T) satisfy $T : X \to \mathrm{Cl}(X)$, and let (X,T) be $\mathcal{J}$-admissible. If there exists the family $\Lambda = \{\lambda_\alpha \in (0,1), \alpha \in \mathcal{A}\}$ such that

$$\forall_{\alpha\in\mathcal{A}} \forall_{x,y\in X} \bigl\{ H^{\mathcal{J}}_\alpha\bigl(T(x), T(y)\bigr) \preceq_H \lambda_\alpha J_\alpha(x,y)\bigr\},$$

then we say that (X,T) is $\mathcal{H}^{\mathcal{J}}_\Lambda$*-contractive*.

(v) Let $E \subseteq X$, $E \neq \varnothing$. The map $F : E \to H \cup \{+\infty\}$ is *lower semicontinuous on E with respect to X* (written: *F is* (E,X)*-lsc* when $E \neq X$ and *F is lsc* when $E = X$) if the set $\{y \in E : F(y) \preceq_H c\}$ is a closed subset in X for each $c \in H$. Equivalently, for each $x_0 \in E$,

$$F(x_0) \preceq_H \liminf_{x\to x_0, x\in X} F(x).$$

(vi) We say that the family $\mathcal{J}$ is *continuous* in X if for each $x_0 \in X$ and for each sequence $(x_m : m \in \mathbb{N})$ in X converging to x_0, we have

$$\forall_{\alpha\in\mathcal{A}} \Bigl\{ \lim_{m\to\infty} J_\alpha(x_m, x_0) = \lim_{m\to\infty} J_\alpha(x_0, x_m) = 0 \Bigr\}.$$

If $\mathcal{J} = \mathcal{P}$, then $\mathcal{J}$ is continuous in X.

3 Statement of result

Let (X, T) be a set-valued dynamic system. By $\mathrm{Fix}(T)$ and $\mathrm{End}(T)$ we denote the sets of all *fixed points* and *endpoints* of T, respectively, *i.e.*, $\mathrm{Fix}(T) = \{w \in X : w \in T(w)\}$ and $\mathrm{End}(T) = \{w \in X : \{w\} = T(w)\}$. A *dynamic process* or a *trajectory starting at* $w_0 \in X$ or a *motion* of the system (X, T) at w_0 is a sequence $(w_m : m \in \{0\} \cup \mathbb{N})$ defined by $w_m \in T(w_{m-1})$ for $m \in \mathbb{N}$ (see, Aubin-Siegel [3] and Yuan [47]).

The aim of this paper is to prove the following fixed point and endpoint existence general result of Nadler type.

Theorem 3.1 (i) *Assume that*:

(A1) *L is an ordered Banach space with a regular solid cone H*;

(A2) *$(X, \mathcal{P})$ is a Hausdorff sequentially complete cone uniform space with a cone H*;

(A3) *$\mathcal{J} = \{J_\alpha : X^2 \to L, \alpha \in \mathcal{A}\}$ is a $\mathcal{J}$-family on X such that $X^0_{\mathcal{J}} \neq \emptyset$*;

(A4) *The set-valued dynamic system (X, T) satisfies $T : X \to \mathrm{Cl}(X)$ and is $\mathcal{J}$-admissible*;

(A5) *There exists the family $\Lambda = \{\lambda_\alpha \in (0, 1), \alpha \in \mathcal{A}\}$ such that (X, T) is $\mathcal{H}^{\mathcal{J}}_{\Lambda}$-contractive*;

(A6) *For each $x \in X$, the set $Q_{\mathcal{J};T}(x)$ is of the form*:

$$Q_{\mathcal{J};T}(x) = \bigl\{y \in T(x) \cap X^0_{\mathcal{J}} : \forall_{\alpha \in \mathcal{A}}\bigl\{J_\alpha\bigl(y, T(y)\bigr) + (\gamma_\alpha - \lambda_\alpha)J_\alpha(x, y) \preceq_H J_\alpha\bigl(x, T(x)\bigr)\bigr\}\bigr\},$$

where the family $\Gamma = \{\gamma_\alpha \in (0, 1), \alpha \in \mathcal{A}\}$ satisfies $\forall_{\alpha \in \mathcal{A}}\{\lambda_\alpha < \gamma_\alpha\}$;

(A7) *For each $x \in X^0_{\mathcal{J}}$, the set $Q_{\mathcal{J};T}(x)$ is a nonempty subset in X; and*

(A8) *For each $x \in X^0_{\mathcal{J}}$, the set $Q_{\mathcal{J};T}(x)$ is a closed subset in X.*

Then the following hold:

(a_1) $\mathrm{Fix}(T) \neq \varnothing$; *and*

(a_2) *For each* $w \in \mathrm{Fix}(T)$, $\forall_{\alpha \in \mathcal{A}}\{J_\alpha(w, w) = 0\}$.

(ii) *Assume*, *in addition*, *that*:

(A9) *For each $x \in X^0_{\mathcal{J}}$, each dynamic process $(w_m : m \in \{0\} \cup \mathbb{N})$ starting at $w_0 = x$ and satisfying $\forall_{m \in \{0\} \cup \mathbb{N}}\{w_{m+1} \in T(w_m)\}$ satisfies $\forall_{m \in \{0\} \cup \mathbb{N}}\{w_{m+1} \in Q_{\mathcal{J};T}(w_m)\}$.*

Then the assertions (a_1) *and* (a_2) *are of the forms*:

(a'_1) $\mathrm{End}(T) \neq \varnothing$; *and*

(a'_2) *For each* $w \in \mathrm{End}(T)$, $\forall_{\alpha \in \mathcal{A}}\{J_\alpha(w, w) = 0\}$.

Remark 3.1 (i) Assume that:

(A10) $\forall_{x \in X^0_{\mathcal{J}}}\{\{y \in T(x) \cap X^0_{\mathcal{J}} : \forall_{\alpha \in \mathcal{A}}\{\gamma_\alpha J_\alpha(x, y) \preceq_H J_\alpha(x, T(x))\}\} \neq \varnothing\}$.
Then (A7) holds.

(ii) Assume that one of the following conditions holds:

(A11) For each $(x, \alpha) \in X^0_{\mathcal{J}} \times \mathcal{A}$, the map

$$J_\alpha\bigl(\cdot, T(\cdot)\bigr) + (\gamma_\alpha - \lambda_\alpha)J_\alpha(x, \cdot) : T(x) \cap X^0_{\mathcal{J}} \to H \cup \{+\infty\}$$

is $(T(x) \cap X^0_{\mathcal{J}}, X)$-lsc;

(A12) The family $\mathcal{J}$ is continuous in X.
Then (A8) holds.

4 Proof of Theorem 3.1

We will use the following fixed point and endpoint existence general result of Caristi type.

Theorem 4.1 ([32, Th. 4.5]) (i) *Assume that*:

(C1) *L is an ordered Banach space with a regular solid cone H*;
(C2) *$(X,\mathcal{P})$ is a Hausdorff sequentially complete cone uniform space with a cone H*;
(C3) *The family $\mathcal{J} = \{J_\alpha : X^2 \to L, \alpha \in \mathcal{A}\}$ is a $\mathcal{J}$-family on X such that $X^0_{\mathcal{J}} \neq \emptyset$*;
(C4) *The family $\Omega = \{\omega_\alpha : X \to H \cup \{+\infty\}, \alpha \in \mathcal{A}\}$ satisfies $D_\Omega = \bigcap_{\alpha\in\mathcal{A}} \operatorname{dom}(\omega_\alpha) \neq \emptyset$*;
(C5) *(X,T) is a set-valued dynamic system*;
(C6) *$\{\varepsilon_\alpha, \alpha \in \mathcal{A}\}$ is a family of finite positive numbers*;
(C7) *For each $x \in X$, the set $Q_{\mathcal{J},\Omega;T}(x)$ is of the form*:

$$Q_{\mathcal{J},\Omega;T}(x) = \bigl\{y \in T(x) \cap X^0_{\mathcal{J}} : \forall_{\alpha\in\mathcal{A}}\bigl\{\omega_\alpha(y) + \varepsilon_\alpha J_\alpha(x,y) \preceq_H \omega_\alpha(x)\bigr\}\bigr\};$$

(C8) *For each $x \in X^0_{\mathcal{J}}$, the set $Q_{\mathcal{J},\Omega;T}(x)$ is a nonempty subset of X; and*
(C9) *For each $x \in X^0_{\mathcal{J}}$, the set $Q_{\mathcal{J},\Omega;T}(x)$ is a closed subset in X.*

Then there exists $w \in D_\Omega \cap X^0_{\mathcal{J}}$ such that

(c) $w \in T(w)$.

(ii) *Assume, in addition, that*:

(C10) *For each $x \in X^0_{\mathcal{J}}$, each dynamic process $(w_m : m \in \{0\} \cup \mathbb{N})$ starting at $w_0 = x$ and satisfying $\forall_{m\in\{0\}\cup\mathbb{N}}\{w_{m+1} \in T(w_m)\}$ satisfies $\forall_{m\in\{0\}\cup\mathbb{N}}\{w_{m+1} \in Q_{\mathcal{J},\Omega;T}(w_m)\}$.*

Then assertion (c) *is of the form*:

(c′) $\{w\} = T(w)$.

Remark 4.1 ([32, Remark 4.6]) (i) A special case of condition (C9) is a condition (C9′) defined by:

(C9′) For each $(x,\alpha) \in X^0_{\mathcal{J}} \times \mathcal{A}$, the map

$$\omega_\alpha(\cdot) + \varepsilon_\alpha J_\alpha(x,\cdot) : T(x) \cap X^0_{\mathcal{J}} \to H \cup \{+\infty\}$$

is $(T(x) \cap X^0_{\mathcal{J}}, X)$-lsc.

(ii) If $\mathcal{J} = \mathcal{P}$, then a special case of condition (C9) is a condition (C9″) defined by:

(C9″) For each $(x,\alpha) \in X \times \mathcal{A}$, the map

$$\omega_\alpha(\cdot) + \varepsilon_\alpha p_\alpha(x,\cdot) : T(x) \to H \cup \{+\infty\}$$

is $(T(x), X)$-lsc.

The proof will be broken into seven steps.

Step 1. *Let $\Omega = \{\omega_\alpha : X \to L, \alpha \in \mathcal{A}\}$ where*

$$\forall_{\alpha\in\mathcal{A}}\forall_{x\in X}\bigl\{\omega_\alpha(x) = J_\alpha\bigl(x, T(x)\bigr)\bigr\}.$$

The following hold:

$$\forall_{\alpha\in\mathcal{A}}\forall_{x\in X}\bigl\{\bigl\{y \in T(x) : \omega_\alpha(y) \preceq_H \lambda_\alpha J_\alpha(x,y)\bigr\} = T(x)\bigr\}; \tag{4.1}$$

$$\forall_{\alpha\in\mathcal{A}}\forall_{x\in X}\bigl\{U_\alpha(x)=\bigl\{y\in T(x):\gamma_\alpha J_\alpha(x,y)\preceq_H \omega_\alpha(x)\bigr\}\neq\varnothing\bigr\};\tag{4.2}$$

$$\forall_{\alpha\in\mathcal{A}}\forall_{x\in X}\bigl\{U_\alpha(x)\subset V_\alpha(x)=\bigl\{y\in T(x):\omega_\alpha(y)+(\gamma_\alpha-\lambda_\alpha)J_\alpha(x,y)\preceq_H \omega_\alpha(x)\bigr\}\bigr\};\tag{4.3}$$

and

$$\forall_{\alpha\in\mathcal{A}}\forall_{x\in X}\forall_{y\in T(x)}\bigl\{0\preceq_H \omega_\alpha(x)-\omega_\alpha(y)\preceq_H \omega_\alpha(x)+\omega_\alpha(y)$$
$$\preceq_H (1+\lambda_\alpha)J_\alpha(x,y)\bigr\}.\tag{4.4}$$

Indeed, by (A4), (A5) and (iii) and (iv) of Definition 2.6, we obtain

$$\forall_{\alpha\in\mathcal{A}}\forall_{x,y\in X}\Bigl\{\sup_{u\in T(x)}J_\alpha\bigl(u,T(y)\bigr)\preceq_H H_\alpha^{\mathcal{J}}\bigl(T(x),T(y)\bigr)\preceq_H \lambda_\alpha J_\alpha(x,y)\Bigr\}.$$

Hence, in particular, for $u=y$, we get

$$\forall_{\alpha\in\mathcal{A}}\forall_{x\in X}\forall_{y\in T(x)}\bigl\{\omega_\alpha(y)\preceq_H \lambda_\alpha J_\alpha(x,y)\bigr\}.$$

This implies (4.1).

Note that

$$\forall_{\alpha\in\mathcal{A}}\forall_{x\in X}\Bigl\{J_\alpha\bigl(x,T(x)\bigr)=\inf_{y\in T(x)}J_\alpha(x,y)\Bigr\}.\tag{4.5}$$

This, by (A6) (recall that $\forall_{\alpha\in\mathcal{A}}\{\gamma_\alpha\in(0,1)\}$), implies (4.2).

By (4.1) and (4.2), we have

$$\forall_{\alpha\in\mathcal{A}}\forall_{x\in X}\bigl\{U_\alpha(x)\subset V_\alpha(x)=\bigl\{y\in T(x):(\gamma_\alpha-\lambda_\alpha)J_\alpha(x,y)$$
$$\preceq_H J_\alpha\bigl(x,T(x)\bigr)-J_\alpha\bigl(y,T(y)\bigr)\bigr\}\bigr\},$$

i.e., (4.3) holds.

By (4.3) and ($\mathcal{J}$1),

$$\forall_{\alpha\in\mathcal{A}}\forall_{x\in X}\forall_{y\in T(x)}\bigl\{0\preceq_H \omega_\alpha(x)-\omega_\alpha(y)\bigr\}$$

and, by (4.5) and (4.1), we have

$$\forall_{\alpha\in\mathcal{A}}\forall_{x\in X}\forall_{y\in T(x)}\bigl\{\omega_\alpha(x)-\omega_\alpha(y)\preceq_H \omega_\alpha(x)+\omega_\alpha(y)$$
$$=J_\alpha\bigl(x,T(x)\bigr)+J_\alpha\bigl(y,T(y)\bigr)\preceq_H (1+\lambda_\alpha)J_\alpha(x,y)\bigr\}.$$

Consequently, (4.4) holds.

Step 2. *The family* Ω *defined in Step* 1 *satisfies* (C4).

Indeed, by (4.1),

$$\forall_{x\in X}\bigl\{\bigl\{y\in T(x):\forall_{\alpha\in\mathcal{A}}\bigl\{\omega_\alpha(y)\preceq_H \lambda_\alpha J_\alpha(x,y)\bigr\}\bigr\}=T(x)\bigr\}.$$

Also, by Definition 2.5, $\mathcal{J}=\{J_\alpha:X^2\to L,\alpha\in\mathcal{A}\}$, ($\mathcal{J}$1) holds and, by (A4), $\forall_{x\in X}\{\varnothing\neq T(x)\}$. Hence, we conclude that $\forall_{x\in X}\{\varnothing\neq T(x)\subset D_\Omega\}$.

Step 3. *Assumptions* (C8) *and* (C9) *hold where* $\forall_{\alpha\in\mathcal{A}}\{\varepsilon_\alpha = \gamma_\alpha - \lambda_\alpha\}$ *and* Ω *is defined in Step* 1.

By (4.2) and (4.3), in particular,

$$\forall_{x\in X^0_{\mathcal{J}}}\forall_{\alpha\in\mathcal{A}}\bigl\{\varnothing \neq \bigl\{y\in T(x): \omega_\alpha(y) + (\gamma_\alpha - \lambda_\alpha)J_\alpha(x,y) \preceq_H \omega_\alpha(x)\bigr\}\bigr\}$$

and, by (A7) and (A8), the following property concerning intersection of these sets holds: for each $x \in X^0_{\mathcal{J}}$,

$$\bigl\{y\in T(x)\cap X^0_{\mathcal{J}} : \forall_{\alpha\in\mathcal{A}}\bigl\{\omega_\alpha(y) + (\gamma_\alpha - \lambda_\alpha)J_\alpha(x,y) \preceq_H \omega_\alpha(x)\bigr\}\bigr\} = Q_{\mathcal{J},\Omega;T}(x)$$

is a nonempty closed subset in X.

Step 4. *The assertions of Theorem* 3.1 *hold*.

This follows from Assumptions (A1)-(A9), Steps 1-3, definition of $X^0_{\mathcal{J}}$ and Theorem 4.1.

Step 5. *Assumption* (A10) *implies* (A7).

Indeed, denote

$$\forall_{x\in X}\left\{U_{\mathcal{J}}(x) = \bigcap_{\alpha\in\mathcal{A}} U_\alpha(x)\right\}$$

and

$$\forall_{x\in X}\left\{V_{\mathcal{J}}(x) = \bigcap_{\alpha\in\mathcal{A}} V_\alpha(x)\right\}.$$

By (4.2) and (4.3),

$$\forall_{x\in X^0_{\mathcal{J}}}\bigl\{U_{\mathcal{J}}(x)\cap X^0_{\mathcal{J}} \subset V_{\mathcal{J}}(x)\cap X^0_{\mathcal{J}} \subset Q_{\mathcal{J};T}(x)\bigr\}.$$

Hence, we conclude that for each $x \in X^0_{\mathcal{J}}$, the set $Q_{\mathcal{J};T}(x)$ is nonempty whenever $\forall_{x\in X^0_{\mathcal{J}}}\{U_{\mathcal{J}}(x)\cap X^0_{\mathcal{J}} \neq \varnothing\}$.

Step 6. *Assumption* (A11) *implies* (A8).

This follows from Remark 4.1(i)

Step 7. *Assumption* (A12) *implies* (A8).

Let x_0 be arbitrary and fixed, and let a sequence $(x_m : m\in\mathbb{N})$ in X be convergent to x_0, *i.e.*, let $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} p_\alpha(x_0,x_m) = 0\}$ (see Definition 2.2 and Theorem 2.1). If $m\in\mathbb{N}$, $v\in T(x_m)$ and $\alpha\in\mathcal{A}$ are arbitrary and fixed, then by ($\mathcal{J}$1),

$$\omega_\alpha(x_0) = J_\alpha\bigl(x_0, T(x_0)\bigr) \preceq_H J_\alpha(x_0,x_m) + J_\alpha(x_m,v) + J_\alpha\bigl(v, T(x_0)\bigr).$$

Since $v\in T(x_m)$ and T satisfy (A5), this implies

$$\begin{aligned}\omega_\alpha(x_0) &\preceq_H J_\alpha(x_0,x_m) + J_\alpha\bigl(x_m, T(x_m)\bigr) + \sup_{v\in T(x_m)} J_\alpha\bigl(v, T(x_0)\bigr)\\ &\preceq_H J_\alpha(x_0,x_m) + J_\alpha\bigl(x_m, T(x_m)\bigr) + H^{\mathcal{J}}_\alpha\bigl(T(x_m), T(x_0)\bigr)\\ &\preceq_H J_\alpha(x_0,x_m) + \omega_\alpha(x_m) + \lambda_\alpha J_\alpha(x_m,x_0),\end{aligned}$$

that is,

$$0 \preceq_H J_\alpha(x_0, x_m) + \omega_\alpha(x_m) + \lambda_\alpha J_\alpha(x_m, x_0) - \omega_\alpha(x_0).$$

Hence, by (A4), since H is closed, using the fact that $\mathcal{J}$ is continuous and taking the limit as $m \to \infty$, we get

$$0 \preceq_H \liminf_{m\to\infty} \omega_\alpha(x_m) - \omega_\alpha(x_0).$$

Therefore, for each $\alpha \in \mathcal{A}$,

$$\omega_\alpha(x_0) \preceq_H \liminf_{m\to\infty} \omega_\alpha(x_m),$$

i.e., ω_α is lsc in X. Moreover, if $m \in \mathbb{N}$, $x \in X$ and $\alpha \in \mathcal{A}$ are arbitrary and fixed, then by ($\mathcal{J}$1),

$$J_\alpha(x, x_0) \preceq_H J_\alpha(x, x_m) + J_\alpha(x_m, x_0),$$

that is,

$$0 \preceq_H J_\alpha(x, x_m) + J_\alpha(x_m, x_0) - J_\alpha(x, x_0).$$

Since H is closed and $\mathcal{J}$ is continuous, this implies

$$0 \preceq_H \liminf_{m\to\infty} J_\alpha(x, x_m) - J_\alpha(x, x_0),$$

that is, for each $(x, \alpha) \in X \times \mathcal{A}$, the map $J_\alpha(x, \cdot)$ is lsc in X. Hence, in particular, we conclude that for each $(x, \alpha) \in X^0_{\mathcal{J}} \times \mathcal{A}$, the map

$$\omega_\alpha(\cdot) + (\gamma_\alpha - \lambda_\alpha) J_\alpha(x, \cdot) : T(x) \cap X^0_{\mathcal{J}} \to H \cup \{+\infty\}$$

is $(T(x) \cap X^0_{\mathcal{J}}, X)$-lsc, that is, (C9′) holds.

5 Remarks, examples and comparisons

Remark 5.1 Examples 5.1 and 5.2 illustrate a fixed point version and an endpoint version of Theorem 3.1, respectively, in cone metric spaces with $\mathcal{J}$-family where $\mathcal{J} = \{J\}$ and $J \neq p$.

Example 5.1 If

$$X = \{\mathbf{N} = (n, n) : n \in \{1, 2, 3, 4, 5, 6\}\} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}\},$$

$L = \mathbb{R}^2$, $H = \{(x, y) \in L : x, y \geq 0\} \subset \mathbb{R}^2$ and, for each $\beta > 0$, $p : X^2 \to L$ is defined by the formula

$$p(\mathbf{N}, \mathbf{M}) = (|n - m|, \beta|n - m|), \quad \mathbf{N} = (n, n), \mathbf{M} = (m, m) \in X,$$

then $(X, \mathcal{P})$, $\mathcal{P} = \{p\}$ is a cone metric space; let in the sequel $\beta = 2$.

Let $T : X \to \mathrm{Cl}(X)$ be of the form:

$$T(\mathbf{N}) = \begin{cases} \{\mathbf{1},\mathbf{2}\} & \text{if } \mathbf{N} \in X \setminus \{\mathbf{6}\}, \\ \{\mathbf{4},\mathbf{5}\} & \text{if } \mathbf{N} = \mathbf{6}. \end{cases}$$

Let $W = \{\mathbf{1},\mathbf{2},\mathbf{4},\mathbf{5}\}$, and let $J : X^2 \to L$ be of the form:

$$J(\mathbf{N},\mathbf{M}) = \begin{cases} p(\mathbf{N},\mathbf{M}) & \text{if } \{\mathbf{N},\mathbf{M}\} \cap W = \{\mathbf{N},\mathbf{M}\}, \\ (8,8) = \mathbf{8} & \text{if } \{\mathbf{N},\mathbf{M}\} \cap W \neq \{\mathbf{N},\mathbf{M}\}, \end{cases}$$

$\mathbf{N},\mathbf{M} \in X$. Clearly, $\mathcal{J} = \{J\}$ is a $\mathcal{J}$-family on X (see [32, Ex. 5.1]).

We observe that $X_J^0 = \{\mathbf{1},\mathbf{2},\mathbf{4},\mathbf{5}\} \neq \emptyset$.

We show that (X,T) is $\mathcal{J}$-admissible and $\mathcal{H}^J_{3/4}$-contractive on X where

$$\forall_{A,B \in \mathrm{Cl}(X)} \Big\{ H^J(A,B) = \max\Big\{ \sup_{\mathbf{N}\in A} J(\mathbf{N},B), \sup_{\mathbf{M}\in B} J(\mathbf{M},A) \Big\} \Big\}.$$

Indeed, let $\lambda = 3/4$, and let $\mathbf{N},\mathbf{M} \in X$ be arbitrary and fixed.

We consider three cases:

Case 1. If $\mathbf{N},\mathbf{M} \in X \setminus \{\mathbf{6}\}$, then by definition of T, we have that $T(\mathbf{N}) = T(\mathbf{M}) = \{\mathbf{1},\mathbf{2}\}$ and

$$H^J\big(T(\mathbf{N}),T(\mathbf{M})\big) = (0,0) = \mathbf{0} \preceq_H (3/4)J(\mathbf{N},\mathbf{M}) = \lambda J(\mathbf{N},\mathbf{M}).$$

Case 2. If $\mathbf{N} \in X \setminus \{\mathbf{6}\}$ and $\mathbf{M} = \mathbf{6}$, then by definition of T, $T(\mathbf{N}) = \{\mathbf{1},\mathbf{2}\}$ and $T(\mathbf{M}) = \{\mathbf{4},\mathbf{5}\}$. Hence, by definition of J, we calculate:

(i) $J(\mathbf{1},T(\mathbf{M})) = p(\mathbf{1},\{\mathbf{4},\mathbf{5}\}) = (3,6)$, $J(\mathbf{2},T(\mathbf{M})) = p(\mathbf{2},\{\mathbf{4},\mathbf{5}\}) = (2,4)$ and

$$\sup\big\{J\big(\mathbf{U},T(\mathbf{M})\big) : \mathbf{U} \in T(\mathbf{N})\big\} = (3,6);$$

(ii) $J(\mathbf{4},T(\mathbf{N})) = p(\mathbf{4},\{\mathbf{1},\mathbf{2}\}) = (2,4)$, $J(\mathbf{5},T(\mathbf{N})) = p(\mathbf{5},\{\mathbf{1},\mathbf{2}\}) = (3,6)$ and

$$\sup\big\{J\big(\mathbf{V},T(\mathbf{N})\big) : \mathbf{V} \in T(\mathbf{M})\big\} = (3,6);$$

(iii) By (i) and (ii),

$$\begin{aligned} H^J\big(T(\mathbf{N}),T(\mathbf{M})\big) &= \max\big\{\sup\big\{J\big(\mathbf{U},T(\mathbf{M})\big) : \mathbf{U} \in T(\mathbf{N})\big\}, \\ &\qquad \sup\big\{J\big(\mathbf{V},T(\mathbf{N})\big) : \mathbf{V} \in T(\mathbf{M})\big\}\big\} = (3,6). \end{aligned}$$

Consequently,

$$H^J\big(T(\mathbf{N}),T(\mathbf{M})\big) = (3,6) \preceq_H \mathbf{6} = (3/4)\cdot\mathbf{8} = \lambda J(\mathbf{N},\mathbf{M})$$

for $\mathbf{N} \in X \setminus \{\mathbf{6}\}$ and $\mathbf{M} = \mathbf{6}$.

Case 3. If $\mathbf{N} = \mathbf{6}$ and $\mathbf{M} \in X \setminus \{\mathbf{6}\}$, then by analogous considerations as in Case 2, we get

$$H^J\big(T(\mathbf{N}),T(\mathbf{M})\big) = (3,6) \preceq_H \mathbf{6} = (3/4)\cdot\mathbf{8} = \lambda J(\mathbf{N},\mathbf{M}).$$

Thus, T is $\mathcal{J}$-admissible and $\mathcal{H}^J_{3/4}$-contractive on X.

Let now $\gamma = 7/8$. Then for each $\mathbf{N} \in X^0_J = \{\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{5}\}$, we have $T(\mathbf{N}) = \{\mathbf{1}, \mathbf{2}\}$ and using the fact that $T(X^0_J) \subset X^0_J$, we obtain

$$\begin{aligned} Q_{J;T}(\mathbf{N}) &= \bigl\{\mathbf{M} \in \{\mathbf{1}, \mathbf{2}\} : J\bigl(\mathbf{M}, T(\mathbf{M})\bigr) + (1/8)J(\mathbf{N}, \mathbf{M}) \preceq_H J\bigl(\mathbf{N}, T(\mathbf{N})\bigr)\bigr\} \\ &= \bigl\{\mathbf{M} \in \{\mathbf{1}, \mathbf{2}\} : (1/8)p(\mathbf{N}, \mathbf{M}) \preceq_H p\bigl(\mathbf{N}, \{\mathbf{1}, \mathbf{2}\}\bigr)\bigr\}. \end{aligned}$$

This implies that

$$\begin{aligned} Q_{J;T}(\mathbf{1}) &= \bigl\{\mathbf{M} \in \{\mathbf{1}, \mathbf{2}\} : (1/8)p(\mathbf{1}, \mathbf{M}) \preceq_H \mathbf{0}\bigr\} = \{\mathbf{1}\}, \\ Q_{J;T}(\mathbf{2}) &= \bigl\{\mathbf{M} \in \{\mathbf{1}, \mathbf{2}\} : (1/8)p(\mathbf{2}, \mathbf{M}) \preceq_H \mathbf{0}\bigr\} = \{\mathbf{2}\} \end{aligned}$$

and

$$Q_{J;T}(\mathbf{N}) = \bigl\{\mathbf{M} \in \{\mathbf{1}, \mathbf{2}\} : (1/8)p(\mathbf{N}, \mathbf{M}) \preceq_H (n-2, 2n-4)\bigr\} = \{\mathbf{1}, \mathbf{2}\}$$

for $\mathbf{N} = (n, n) \in \{\mathbf{4}, \mathbf{5}\}$.

Assumptions (A1)-(A8) of Theorem 3.1 hold, $\mathrm{Fix}(T) = \{(1,1), (2,2)\}$ and $J((1,1),(1,1)) = J((2,2),(2,2)) = 0$.

Example 5.2 Let X, W, J, λ and γ be such as in Example 5.1, and let $T : X \to \mathrm{Cl}(X)$ be of the form:

$$T(\mathbf{N}) = \begin{cases} \{\mathbf{1}\} & \text{if } \mathbf{N} \in \{\mathbf{1}, \mathbf{3}, \mathbf{5}\}, \\ \{\mathbf{2}\} & \text{if } \mathbf{N} \in \{\mathbf{2}, \mathbf{4}\}, \\ \{\mathbf{4}, \mathbf{5}\} & \text{if } \mathbf{N} = \{\mathbf{6}\}. \end{cases}$$

Then $X^0_J = \{\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{5}\}$ and

$$Q_{J;T}(\mathbf{N}) = \bigl\{\mathbf{M} \in T(\mathbf{N}) : J\bigl(\mathbf{M}, T(\mathbf{M})\bigr) + (1/8)J(\mathbf{N}, \mathbf{M}) \preceq_H J\bigl(\mathbf{N}, T(\mathbf{N})\bigr)\bigr\}$$

for $\mathbf{N} \in X^0_J$ since $T(X^0_J) \subset X^0_J$. Hence:

$$\begin{aligned} Q_{J;T}(\mathbf{1}) &= \bigl\{\mathbf{M} \in \{\mathbf{1}\} : J(\mathbf{1}, \mathbf{1}) + (1/8)J(\mathbf{1}, \mathbf{1}) \preceq_H J(\mathbf{1}, \mathbf{1})\bigr\} = \{\mathbf{1}\}; \\ Q_{J;T}(\mathbf{2}) &= \bigl\{\mathbf{M} \in \{\mathbf{2}\} : J(\mathbf{2}, \mathbf{2}) + (1/8)J(\mathbf{2}, \mathbf{2}) \preceq_H J(\mathbf{2}, \mathbf{2})\bigr\} = \{\mathbf{2}\}; \\ Q_{J;T}(\mathbf{4}) &= \bigl\{\mathbf{M} \in \{\mathbf{2}\} : J(\mathbf{2}, \mathbf{2}) + (1/8)J(\mathbf{4}, \mathbf{2}) \preceq_H J(\mathbf{4}, \mathbf{2})\bigr\} = \{\mathbf{2}\}; \end{aligned}$$

and

$$Q_{J;T}(\mathbf{5}) = \bigl\{\mathbf{M} \in \{\mathbf{1}\} : J(\mathbf{1}, \mathbf{1}) + (1/8)J(\mathbf{5}, \mathbf{1}) \preceq_H J(\mathbf{5}, \mathbf{1})\bigr\} = \{\mathbf{1}\}.$$

Assumptions (A1)-(A9) of Theorem 3.1 hold, $\mathrm{End}(T) = \{(1,1), (2,2)\}$ and $J((1,1),(1,1)) = J((2,2),(2,2)) = 0$.

Remark 5.2 In Example 5.3, we show that in our concept of $\mathcal{H}_{\Lambda}^{\mathcal{J}}$-contractive set-valued dynamic systems, the existence of $\mathcal{J}$-family such that $\mathcal{J} \neq \mathcal{D}$ is essential; from Example 5.3, it follows that for maps defined in Examples 5.1 and 5.2, we cannot use Theorem 3.1 when $\mathcal{J} = \{p\}$.

Example 5.3 (a) Let X and T be such as in Example 5.1. We observe that for each $\lambda \in (0,1)$, T is not $\mathcal{H}_{\lambda}^{p}$-contractive on X.

Otherwise, $J = p$, $X_p^0 = X$ and

$$\exists_{\lambda\in(0,1)}\forall_{\mathbf{N},\mathbf{M}\in X}\bigl\{H^p\bigl(T(\mathbf{N}),T(\mathbf{M})\bigr)\leq\lambda p(\mathbf{N},\mathbf{M})\bigr\}.$$

However, for $\mathbf{N}_0 = \mathbf{3}$ and $\mathbf{M}_0 = \mathbf{6}$ from X, we obtain:

(i) $T(\mathbf{N}_0) = \{\mathbf{1},\mathbf{2}\}$ and $T(\mathbf{M}_0) = \{\mathbf{4},\mathbf{5}\}$;

(ii) $p(\mathbf{1},T(\mathbf{M}_0)) = p(\mathbf{1},\{\mathbf{4},\mathbf{5}\}) = (3,6)$, $p(\mathbf{2},T(\mathbf{M}_0)) = p(\mathbf{2},\{\mathbf{4},\mathbf{5}\}) = (2,4)$ and

$$\sup\bigl\{p\bigl(\mathbf{U},T(\mathbf{M}_0)\bigr):\mathbf{U}\in T(\mathbf{N}_0)\bigr\} = (3,6);$$

(iii) $p(\mathbf{4},T(\mathbf{N}_0)) = p(\mathbf{4},\{\mathbf{1},\mathbf{2}\}) = (2,4)$, $p(\mathbf{5},T(\mathbf{N}_0)) = p(\mathbf{5},\{\mathbf{1},\mathbf{2}\}) = (3,6)$ and

$$\sup\bigl\{p\bigl(\mathbf{V},T(\mathbf{N}_0)\bigr):\mathbf{V}\in T(\mathbf{M}_0)\bigr\} = (3,6);$$

(iv) By (i)-(iii),

$$\begin{aligned} H^p\bigl(T(\mathbf{N}_0),T(\mathbf{M}_0)\bigr) = \max\bigl\{&\sup\bigl\{p\bigl(\mathbf{U},T(\mathbf{M}_0)\bigr):\mathbf{U}\in T(\mathbf{N}_0)\bigr\},\\ &\sup\bigl\{p\bigl(\mathbf{V},T(\mathbf{N}_0)\bigr):\mathbf{V}\in T(\mathbf{M}_0)\bigr\}\bigr\} = (3,6). \end{aligned}$$

Consequently, for each $\lambda \in (0,1)$,

$$\begin{aligned} (3,6) &= H^p\bigl(T(\mathbf{N}_0),T(\mathbf{M}_0)\bigr) \preceq_H \lambda p(\mathbf{N}_0,\mathbf{M}_0) \prec_H p(\mathbf{N}_0,\mathbf{M}_0)\\ &= p(\mathbf{3},\mathbf{6}) = (3,6). \end{aligned}$$

It is absurd.

(b) Let X and T be such as in Example 5.2. By similar argumentation as in (a), we observe that for each $\lambda \in (0,1)$, T is not $\mathcal{H}_{\lambda}^{p}$-contractive on X.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors have equitably contributed in obtaining the new results presented in this article. All authors read and approved the final manuscript.

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